

A symmetry property for polyharmonic functions vanishing on equidistant hyperplanes

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Abstract

Let $u(t, y)$ be a polyharmonic function of order N defined on the strip $(a, b) \times \mathbb{R}^d$ satisfying the growth condition

$$\sup_{t \in K} |u(t, y)| \leq o\left(|y|^{(1-d)/2} e^{\frac{\pi}{c}|y|}\right) \quad (1)$$

for $|y| \rightarrow \infty$ and any compact subinterval K of (a, b) , and suppose that $u(t, y)$ vanishes on $2N - 1$ equidistant hyperplanes of the form $\{t_j\} \times \mathbb{R}^d$ for $t_j = t_0 + jc \in (a, b)$ and $j = -(N - 1), \dots, N - 1$. Then it is shown that $u(t, y)$ is odd at t_0 , i.e. that $u(t_0 + t, y) = -u(t_0 - t, y)$ for $y \in \mathbb{R}^d$. The second main result states that u is identically zero provided that u satisfies (1) and vanishes on $2N$ equidistant hyperplanes with distance c .

1 Introduction

A function $f : G \rightarrow \mathbb{C}$ defined on a domain G in the euclidean space \mathbb{R}^d is called *polyharmonic of order N* if f is $2N$ times continuously differentiable and

$$\Delta^N f(x) = 0$$

for all $x \in G$, where $\Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_d^2}$ is the Laplace operator and Δ^N the N -th iterate of Δ . Polyharmonic functions play an important role in pure and applied mathematics, and they have been studied extensively in [2], [3], [21]. In [11] polyharmonic functions are studied in the context of boundary value problems for partial differential operators of higher order. In applied mathematics they are important for multivariate interpolation and spline theory [15], [16], [17], [20], for constructing new types of cubature formulae [18], or for constructing wavelets and subdivision schemes [6], [7]. Polyharmonic functions of order 2 are called *biharmonic functions* and they play an eminent role in elasticity theory, cf. the references in [13].

An important tool in the theory of polyharmonic functions is the Almansi decomposition which was proven in 1899 in [1]: for a polyharmonic function f of order

N defined on a star domain G there exist harmonic functions $h_k : G \rightarrow \mathbb{C}$ such that

$$f(x) = \sum_{k=0}^{N-1} |x|^{2k} h_k(x), \quad (2)$$

see e.g. [2]. A simple consequence is the following result: if G is a ball of radius $R > 0$ with center 0 and if the polyharmonic function f vanishes on the concentric spheres $\{x \in \mathbb{R}^d : |x| = t_j\}$ for given radii $0 < t_1 < \dots < t_N < R$ then f is identically zero. In other words: a polyharmonic function of order N is completely determined by its values on N concentric spheres. This result was generalized in [23] replacing concentric spheres by the boundaries of ellipsoids in arbitrary position answering positively a question in [14].

In this paper we consider a similar question for polyharmonic functions $u(t, y)$ of order N defined on the strip

$$(a, b) \times \mathbb{R}^d = \{(t, y) : t \in (a, b) \text{ and } y \in \mathbb{R}^d\}.$$

We ask under which conditions is it true that

- (*) a polyharmonic function of order N vanishing on $2N$ hyperplanes of the form $\{t_j\} \times \mathbb{R}^d$ for real numbers $t_1 < \dots < t_{2N}$ in the open interval (a, b) is identically zero?

In passing we mention that this question arises naturally in the context of polyharmonic interpolation for data functions given on the hyperplanes $\{t_j\} \times \mathbb{R}^d$ for $j = 1, \dots, 2N$, see [19]. The case $N = 1$ and $d = 2$ already shows that one needs additional assumptions for a positive answer: the harmonic function $u(t, y) = \sin t \cdot e^y$ vanishes on two parallel lines but it is not the zero function. On the other hand, it is well known that a harmonic function vanishing on two parallel lines is identically zero if it has exponential growth of order less than 1. Thus one can expect only a positive answer if $u(t, y)$ satisfies certain growth estimates.

We shall show that statement (*) is true provided that (i) the points t_1, \dots, t_{2N} are equidistant, i.e. $t_j = t_1 + (j - 1)c$ for $j = 1, \dots, 2N$, and (ii) $u(t, y)$ satisfies the growth estimate

$$\sup_{t \in [t_1, t_{2N}]} |u(t, y)| \leq o\left(|y|^{(1-d)/2} e^{\frac{\pi}{c}|y|}\right)$$

for $|y| \rightarrow \infty$. We believe that the result is true without the assumption (i) of equidistant points but we have been unable to provide a proof.

The proof of this result is based on a new symmetry property for polyharmonic functions which is interesting in its own right. Let us say that a function $u : (a, b) \times \mathbb{R}^d \rightarrow \mathbb{C}$ is *odd* at $t_0 \in (a, b)$ if

$$u(t_0 + t, y) = -u(t_0 - t, y) \quad (3)$$

for all $t \in (t_0 - c, t_0 + c)$ and $y \in \mathbb{R}^d$ where $c := \min\{b - t_0, a - t_0\}$. We shall use the convention that $u(t, y)$ is odd if it is odd at the point $t_0 = 0$. Note that

necessarily $u(t_0, y) = 0$ for all $y \in \mathbb{R}^d$ if u is odd at t_0 . For a *harmonic* function $h(t, y)$ the Schwarz reflection principle shows that the converse is also true: h is odd at t_0 if and only if $h(t_0, y) = 0$ for all $y \in \mathbb{R}^d$. Simple examples show that this is not true for biharmonic functions.

We prove the following symmetry principle: Let $u : (a, b) \times \mathbb{R}^d \rightarrow \mathbb{C}$ be biharmonic and suppose that there exists $t_1 \in (a, b)$ and $c > 0$ such that $a < t_1 - c < t_1 + c < b$ and

$$u(t_1, y) = 0 \text{ and } u(t_1 + c, y) = -u(t_1 - c, y)$$

for all $y \in \mathbb{R}^d$. If

$$\sup_{t \in [t_1 - c, t_1 + c]} |u(t, y)| \leq o\left(|y|^{(1-d)/2} e^{\frac{\pi}{c}|y|}\right) \text{ for } |y| \rightarrow \infty$$

then u is odd at t_1 . An analogous statement holds for polyharmonic functions of order N .

Let us now outline the paper: in Section 2 we briefly investigate the Fourier series of a polyharmonic function $h : (-\pi - \delta, \pi + \delta) \times \mathbb{R}^d \rightarrow \mathbb{C}$ of order N . If $h(t, y)$ is odd at $t = 0$ and $t = \pm\pi$ then it is shown that the Fourier coefficients

$$a_k(y) = \frac{1}{\pi} \int_{-\pi}^{\pi} h(t, y) \sin ktdt$$

satisfy the equation $(\Delta_y - k^2)^N a_k(y) = 0$. If in addition $h(t, y)$ satisfies the growth assumption

$$\sup_{t \in [-\pi, \pi]} |t \cdot (\pi^2 - t^2) h(t, y)| \leq o\left(|y|^{(1-d)/2} e^{|y|}\right)$$

for $|y| \rightarrow \infty$ then one can prove that h is identically zero using results of Vekua, Rellich and Friedman. This result will play a fundamental role in the next sections.

In Section 3 the symmetry principle is proved for biharmonic functions. In Section 4 the general case is discussed which follows the same lines but is technically more involved.

In Section 2 we encountered polyharmonic functions which are odd for two different points. In the appendix we prove by elementary means that a polyharmonic function $u : (a, b) \times \mathbb{R}^d \rightarrow \mathbb{C}$ possesses a polyharmonic 2δ -periodic extension \tilde{u} to $(-\infty, \infty) \times \mathbb{R}^d$ provided that $u(t, y)$ is odd at two different points $t_1 < t_2 \in (a, b)$ so that $a < t_1 - \delta$ and $t_2 + \delta < b$ for $\delta := t_2 - t_1$.

2 Fourier series of polyharmonic functions on a strip

Harmonic functions on the strip in \mathbb{R}^2 or on half spaces have been extensively studied in [4], [5], [10], [12] or [25]. We need extensions of these results to polyharmonic functions of order N and for general dimension d .

Theorem 1 *Let $\delta > 0$ and assume that $h : (-\pi - \delta, \pi + \delta) \times \mathbb{R}^d \rightarrow \mathbb{C}$ is a polyharmonic function of order N which is odd at $t = 0$ and $t = \pm\pi$. Then the Fourier*

coefficients $a_k(y)$ defined in (4) of the function $H_y(t) := h(t, y)$ on $[-\pi, \pi]$ for $y \in \mathbb{R}^d$ satisfy the equation

$$(\Delta_y - k^2)^N a_k(y) = 0.$$

Proof. The function $H_y(t) := h(t, y)$ is odd on $[-\pi, \pi]$ for each $y \in \mathbb{R}^d$. Thus the Fourier coefficients of H_y for the basis function $\cos kt$ vanish. Next consider the Fourier coefficients

$$a_k(y) = \frac{1}{\pi} \int_{-\pi}^{\pi} h(t, y) \sin ktdt. \quad (4)$$

Clearly $y \mapsto h(t, y)$ is a C^∞ -function on \mathbb{R}^d , and $t \mapsto \frac{\partial^s}{\partial y_j^s} h(t, y)$ for $s = 1, \dots, N$, is bounded on $[-\pi - \delta/2, \pi + \delta/2]$. Using Lebesgue's dominated convergence theorem we obtain

$$\Delta_y^N a_k(y) = \frac{1}{\pi} \int_{-\pi}^{\pi} \Delta_y^N h(t, y) \cdot \sin ktdt.$$

Since $h(t, y)$ is polyharmonic of order N we know that

$$0 = \left(\frac{d^2}{dt^2} + \Delta_y \right)^N h(t, y) = \Delta_y^N h(t, y) + \sum_{l=1}^N \binom{N}{l} \frac{\partial^{2l}}{\partial t^{2l}} \Delta_y^{N-l} h(t, y).$$

Moreover, $h(t, y)$ is odd at $\pm\pi$ and this obviously implies that $\Delta_y^s h(t, y)$ is odd at $\pm\pi$, so

$$\frac{\partial^{2l}}{\partial t^{2l}} \Delta_y^{N-l} h(\pm\pi, y) = 0.$$

Repeated partial integration shows that

$$\begin{aligned} \Delta_y^N a_k(y) &= - \sum_{l=1}^N \binom{N}{l} (-k^2)^l \frac{1}{\pi} \int_{-\pi}^{\pi} \Delta_y^{N-l} h(t, y) \sin ktdt \\ &= - \sum_{l=1}^N \binom{N}{l} (-k^2)^l \Delta_y^{N-l} a_k(y). \end{aligned}$$

From this the statement is obvious. ■

Remark 2 For $N = 1$ it suffices to require that $h : [-\pi, \pi] \times \mathbb{R}^d \rightarrow \mathbb{C}$ is an odd continuous function which is harmonic on $(-\pi, \pi) \times \mathbb{R}^d$ such that $h(-\pi, y) = h(\pi, y) = 0$ for all $y \in \mathbb{R}^d$. Indeed, by Schwarz's reflection principle one can extend h to a harmonic function on $(-\infty, \infty) \times \mathbb{R}^d$.

Remark 3 The assumption that $h(t, y)$ vanishes for $t = \pm\pi$ is essential: the harmonic function $h(t, y) = t \cdot y$ has the Fourier series

$$\begin{aligned} h(t, y) &= \sum_{k=1}^{\infty} a_k(y) \\ \sin kt \text{ with } a_k(y) &= (-1)^{k+1} \frac{2y}{k}. \end{aligned}$$

Clearly a_k is not a solution to the equation $\Delta_y(a_k) = k^2 a_k$.

Next we need a result which goes back to I. Vekua and F. Rellich in the 1940's for the case $N = 1$, see [24], and which can be found in [9, p. 228] for general N :

Theorem 4 *Let k be a positive number and suppose that $f : \mathbb{R}^d \rightarrow \mathbb{C}$ is a solution of the equation $(\Delta_y - k^2)^N f(y) = 0$. If*

$$|f(y)| = o\left(|y|^{(1-d)/2} e^{k|y|}\right)$$

for $|y| \rightarrow \infty$ then $f(y)$ is identically zero.

Corollary 5 *Let $\delta > 0$ and assume that $h : (-\pi - \delta, \pi + \delta) \times \mathbb{R}^d \rightarrow \mathbb{C}$ is polyharmonic of order N which is odd at $t = 0$ and $t = \pm\pi$. If*

$$\sup_{t \in [-\pi, \pi]} |t \cdot (\pi^2 - t^2) h(t, y)| \leq o\left(|y|^{(1-d)/2} e^{|y|}\right) \quad (5)$$

for $|y| \rightarrow \infty$ then $h(t, y)$ is identically zero.

Proof. By Theorem 1 it suffices to show that the Fourier coefficient $a_k(y)$ defined in (4) is zero. Note that

$$a_k(y) = \frac{1}{\pi} \int_{-\pi}^{\pi} h(t, y) \sin kt dt = \frac{1}{\pi} \int_{-\pi}^{\pi} t (\pi^2 - t^2) \cdot h(t, y) \frac{\sin kt}{t (\pi^2 - t^2)} dt.$$

Since $\sin kt/t(\pi^2 - t^2)$ is bounded on the interval $[-\pi, \pi]$, it is easy to show that assumption (5) leads to the estimate

$$a_k(y) = o\left(|y|^{(1-d)/2} e^{|y|}\right).$$

Theorem 4 completes the proof. ■

3 A symmetry principle for biharmonic functions

Biharmonic functions are difficult to investigate since they behave rather differently from their harmonic peers. Sometimes it is instructive to consider the univariate case: a biharmonic function $f : (a, b) \rightarrow \mathbb{C}$ is just a solution of the equation

$$\frac{d^4}{dx^4} f(x) = 0,$$

thus $f(x)$ is a polynomial of degree 3. It is well known that a polynomial of degree 3 which vanishes in 4 points is identically zero. Moreover a polynomial f satisfying $f(0) = 0$ and $f(-c) = -f(c)$ is odd. In this section we provide analogs of these statements for biharmonic functions.

The following result will be our main tool:

Theorem 6 Suppose that $u(t, y)$ is polyharmonic of order N on $(a, b) \times \mathbb{R}^d$ and let $t_0 \in (a, b)$ such that $u(t_0, y) = 0$ for all $y \in \mathbb{R}^d$. Define $c := \min\{b - t_0, t_0 - a\} > 0$. Then there exists an odd polyharmonic function H_{N-1} of order $N - 1$ defined on $(-c, c) \times \mathbb{R}^d$ such that

$$u(t_0 + t, y) + u(t_0 - t, y) = t \cdot H_{N-1}(t, y).$$

Proof. Define a polyharmonic function \tilde{u} defined on $(a - t_0, b - t_0) \times \mathbb{R}^d$ by $\tilde{u}(t, y) = u(t + t_0, y)$. By a theorem of Almansi (see e.g. [1], [3], [9] or [22]) there exist harmonic functions $h_j : (a - t_0, b - t_0) \times \mathbb{R}^d \rightarrow \mathbb{C}$ such that

$$\tilde{u}(t, y) = \sum_{j=0}^{N-1} t^j \cdot h_j(t, y)$$

for all $(t, y) \in (a - t_0, b - t_0) \times \mathbb{R}^d$. Since $\tilde{u}(0, y) = 0$ it follows that $h_0(0, y) = 0$. Since h_0 is harmonic it follows that $h_0(-t, y) = -h_0(t, y)$. Then we obtain that

$$\tilde{u}(t, y) + \tilde{u}(-t, y) = t \cdot H_1(t, y)$$

for $|t| < \min\{b - t_0, t_0 - a\}$ where

$$H_1(t, y) = \sum_{j=1}^{N-1} t^{j-1} \cdot \left(h_j(t, y) + (-1)^j h_j(-t, y) \right)$$

is polyharmonic of order $\leq N - 1$. The function $H_1(t, y)$ is odd since $t \mapsto u(t_0 + t, y) + u(t_0 - t, y)$ is even. ■

Theorem 7 Let $u : (a, b) \times \mathbb{R}^d \rightarrow \mathbb{C}$ be biharmonic and suppose that there exists $t_1 \in (a, b)$ and $c > 0$ such that $a < t_1 - c < t_1 + c < b$ and

$$u(t_1, y) = 0 \text{ and } u(t_1 + c, y) = -u(t_1 - c, y)$$

for all $y \in \mathbb{R}^d$. If

$$\sup_{t \in [t_1 - c, t_1 + c]} |u(t, y)| \leq o\left(|y|^{(1-d)/2} e^{\frac{\pi}{c}|y|}\right)$$

for $|y| \rightarrow \infty$ then u is odd at $t = t_1$, i.e. that $u(t_1 + t, y) = -u(t_1 - t, y)$ for all $t \in [-c, c]$ and all $y \in \mathbb{R}^d$.

Proof. Define $\tilde{u}(t, y) = u(t_1 + \frac{c}{\pi}t, \frac{c}{\pi}y)$. Then $\tilde{u}(0, y) = 0$ and $\tilde{u}(\pi, y) = -\tilde{u}(-\pi, y)$. By Theorem 6 there exists an odd harmonic function H such that

$$\tilde{u}(t, y) + \tilde{u}(-t, y) = t \cdot H(t, y).$$

It suffices to show that $H(t, y)$ is identically zero. Clearly $H(\pm\pi, y) = 0$. We can now estimate

$$\sup_{t \in [-\pi, \pi]} |t \cdot H(t, y)| \leq 2 \sup_{t \in [-\pi, \pi]} |\tilde{u}(t, y)| = 2 \sup_{t \in [-\pi, \pi]} \left| u\left(t_1 + \frac{c}{\pi}t, \frac{c}{\pi}y\right) \right|.$$

Now the result follows from Corollary 5. ■

Theorem 8 Let $u : (a, b) \times \mathbb{R}^d \rightarrow \mathbb{C}$ be biharmonic and suppose that there exists $t_0 \in (a, b)$ and $c > 0$ such that $a < t_0 < t_0 + 3c < b$ and

$$u(t_0 + jc, y) = 0 \text{ and for } j = 0, 1, 2, 3$$

and for all $y \in \mathbb{R}^d$. If

$$\sup_{t \in [t_0, t_0 + 3c]} |u(t, y)| \leq o\left(|y|^{(1-d)/2} e^{\frac{\pi}{c}|y|}\right)$$

then u is identically zero.

Proof. Put $t_j = t_0 + jc$ for $j = 0, 1, 2, 3$. We apply Theorem 7 to the point $t_0 + c$ (we know that $u(t, y)$ vanishes for $t = t_0$ and $t_0 + c$ and $t_0 + 2c$) and we infer that u is odd at $t_0 + c$. Theorem 7 applied to the point $t_0 + 2c$ shows that u is odd at $t_0 + 2c$. Then it is easy to see that u is as well odd at t_0 . Using a transformation of variables we can apply Corollary 5 and we infer that u must be identical zero. ■

4 A symmetry principle for polyharmonic functions

We want to generalize Theorem 7 to the case of polyharmonic functions $u : (a, b) \times \mathbb{R}^d \rightarrow \mathbb{C}$ of order N which vanish on $2N - 1$ equidistant hyperplanes $\{t_j\} \times \mathbb{R}^d$. By using simple transformations we may assume that $t_j = j\pi$ for $j = 0, \pm 1, \dots, \pm(N - 1)$ and $u(t, y)$ is defined for all $|t| < N - 1 + \delta$.

Theorem 9 Let $\delta > 0$ and $c_N = (N - 1)\pi + \delta$. Let $u : (-c_N, c_N) \times \mathbb{R}^d \rightarrow \mathbb{C}$ be polyharmonic of order N and suppose that

$$u(j\pi, y) = -u(-j\pi, y) \quad \text{for } j = 0, 1, \dots, N - 1$$

for all $y \in \mathbb{R}^d$ and suppose that

$$\sup_{t \in (-c_N, c_N)} |u(t, y)| \leq o\left(|y|^{(1-d)/2} e^{|y|}\right)$$

for $|y| \rightarrow \infty$. Then u is odd.

Proof. We apply Theorem 6 to the point $t_0 = 0$. Then there exists a polyharmonic odd function $H_{N-1}(t, y)$ of order $N - 1$ defined on $(-c_N, c_N) \times \mathbb{R}^d$ such that

$$u(t, y) + u(-t, y) = t \cdot H_{N-1}(t, y)$$

It follows that $H_{N-1}(\pm j\pi, y) = 0$ for $j = 0, \dots, N - 1$. Theorem 6 applied to the function $H_{N-1}(t, y)$ and the point π shows that there exists a polyharmonic odd function $H_{N-2}(t, y)$ of order $N - 2$ defined on $(-c_{N-1}, c_{N-1}) \times \mathbb{R}^d$ such that

$$H_{N-1}(\pi + t, y) + H_{N-1}(\pi - t, y) = t \cdot H_{N-2}(t, y) \quad (6)$$

where $c_{N-1} = c_N - \pi$. Clearly $H_{N-2}(j\pi, y) = 0$ for $j = 0, \dots, N-2$. Having constructed the odd polyharmonic function $H_{N-(j-1)}$ of order $N - (j-1)$ vanishing on $(j\pi, y)$ for $j = 0, \dots, N - (j-1)$ we define the odd polyharmonic function $H_{N-j}(t, y)$ of order $N - j$ by Theorem 6 by the equation

$$H_{N-(j-1)}(\pi + t, y) + H_{N-(j-1)}(\pi - t, y) = t \cdot H_{N-j}(t, y). \quad (7)$$

It follows that $H_{N-1}(j\pi, y) = 0$ for $j = 0, \dots, N-j$. Equation (9) in the next theorem shows that

$$\sup_{t \in [-\pi, \pi]} |t \cdot (\pi^2 - t^2) H_{N-j}(t, y)| = o\left(|y|^{(1-d)/2} e^{|y|}\right). \quad (8)$$

For $j = N-1$ it follows that H_1 is a harmonic function which is odd at 0 and π and satisfies (8). Corollary 5 shows that H_1 is identically 0. It follows from (7) for $j = N-1$ that H_2 is odd at π . Corollary 5 and (8) show that H_2 is zero. Now one can proceed inductively and one obtains that H_1, \dots, H_{N-1} are identically zero, which clearly implies that u is odd. ■

Theorem 10 *Let $u(t, y)$ and $H_{N-j}(t, y)$ as in the proof of Theorem 9 and define*

$$\begin{aligned} w_N(t, y) &:= t \cdot H_{N-1}(t, y) = u(t, y) + u(-t, y) \\ A_j(t) &:= (\pi + t)(2\pi + t) \cdots (\pi j + t). \end{aligned}$$

Then the following identity holds

$$A_j(t) A_j(-t) \cdot t \cdot H_{N-(j-1)}(t, y) = \sum_{l=0}^j p_{j,l}(t) w_N((j-2l)\pi + t, y) \quad (9)$$

where $p_{j,0}(t) = A_j(-t)$ and $p_{j,j}(t) = A_j(t)$, and $p_{j,l}(t)$ are polynomials of degree j for $l = 1, \dots, j-1$ defined by

$$p_{j,l}(t) = \binom{j}{l} \prod_{s=0}^{l-1} [(j-s)\pi - t][(j-s)\pi + t] \cdot \prod_{s=0}^{j-1-2l} ((j-l-s)\pi - t). \quad (10)$$

Proof. We consider at first the case $j = 1$. By (6)

$$t H_{N-2}(t, y) = H_{N-1}(\pi + t, y) + H_{N-1}(\pi - t, y).$$

We multiply this identity by $\pi^2 - t^2$, and we verify the validity of (9) using $A_1(t) = \pi + t$:

$$t(\pi^2 - t^2) H_{N-2}(t, y) = (\pi - t) w_N(\pi + t, y) + (\pi + t) w_N(\pi - t, y).$$

For the general case multiply the equation (7) with $t \cdot A_j(t) A_j(-t)$

$$t^2 \cdot A_j(t) A_j(-t) \cdot H_{N-j}(t, y) \quad (11)$$

$$= t \cdot A_j(t) A_j(-t) H_{N-(j-1)}(\pi + t) + t \cdot A_j(t) A_j(-t) H_{N-(j-1)}(\pi - t). \quad (12)$$

Note that $A_j(t) = (\pi + t) A_{j-1}(\pi + t)$. Since

$$A_{j-1}(-(t + \pi)) = (-t)(\pi - t) \cdots (\pi(j - 2) - t)$$

it follows that

$$(-t) \cdot A_j(-t) = (\pi j - t)(\pi(j - 1) - t) \cdot A_{j-1}(-(t + \pi)). \quad (13)$$

If we define

$$w_{N-j}(t, y) := A_j(t) A_j(-t) \cdot t \cdot H_{N-(j-1)}(t, y)$$

then equation (11) shows that

$$\begin{aligned} t \cdot w_{N-j}(t, y) &= -(\pi j - t)(\pi(j - 1) - t) \cdot w_{N-(j-1)}(\pi + t, y) \\ &\quad + (\pi j + t)(\pi(j - 1) + t) \cdot w_{N-(j-1)}(t - \pi, y). \end{aligned} \quad (14)$$

where we have used that $A_j(-t) = (\pi - t) A_{j-1}(\pi - t)$ and (13) for $-t$, and in the last step the fact that $w_{N-(j-1)}$ is even in t . Unfortunately, this recursion does not give an easy estimate of the function $w_{N-j}(t, y)$ due to the presence of the factor t on the left hand side of the formula.

Now we prove the existence of the representation (9) by induction over j . For $j = 1$ this has been verified and assume that it is true for $j - 1$, namely

$$w_{N-(j-1)}(t, y) = \sum_{l=0}^{j-1} p_{j-1,l}(t) w_N((j - 1 - 2l)\pi + t, y).$$

Thus we obtain

$$\begin{aligned} w_{N-(j-1)}(t + \pi) &= \sum_{l=0}^{j-1} p_{j-1,l}(t + \pi) w_N((j - 2l)\pi + t) \\ w_{N-(j-1)}(t - \pi) &= \sum_{l=0}^{j-1} p_{j-1,l}(t - \pi) w_N(t + (j - 2 - 2l)\pi). \end{aligned}$$

Now (14) shows that

$$t \cdot w_{N-j}(t, y) = \sum_{l=0}^j \tilde{p}_{j,l}(t) w_N((j - 2l)\pi + t, y)$$

where

$$\begin{aligned} \tilde{p}_{j,0}(t) &= -(-\pi j - t)(\pi(j - 1) - t) p_{j-1,0}(t + \pi) \\ \tilde{p}_{j,j}(t) &= -(\pi j + t)(\pi(j - 1) + t) p_{j-1,j-1}(t - \pi) \end{aligned}$$

and for $l = 1, \dots, j - 1$

$$\tilde{p}_{j,l}(t) = -(\pi j - t)(\pi(j - 1) - t) p_{j-1,l}(t + \pi) + (\pi j + t)(\pi(j - 1) + t) p_{j-1,l-1}(t - \pi).$$

Straightforward but tedious calculations (see Appendix 2) show that

$$\frac{\tilde{p}_{j,l}(t)}{t} = p_{j,l}(t).$$

■

Following the proof of Theorem 8 one may derive from Theorem 9 the main result of the paper :

Theorem 11 *Let $u : (a, b) \times \mathbb{R}^d \rightarrow \mathbb{C}$ be polyharmonic of order N and suppose that there exist $t_0 \in (a, b)$ and $c > 0$ such that $a < t_0 < t_0 + (2N - 1)c < b$ and*

$$u(t_0 + jc, y) = 0 \quad \text{for } j = 0, 1, \dots, 2N - 1$$

and for all $y \in \mathbb{R}^d$. If

$$\sup_{t \in [t_0, t_0 + c(2N-1)]} |u(t, y)| \leq o\left(|y|^{(1-d)/2} e^{\frac{\pi}{c}|y|}\right)$$

for $|y| \rightarrow \infty$, then u is identically zero.

5 Appendix 1: Periodic extensions of polyharmonic functions

A reflection law for biharmonic functions at a hyperplane $\{t_0\} \times \mathbb{R}^d$ was introduced by Poritsky and extended by Duffin (see [22]). Unfortunately it is required in these results that not only $u(t_0, y)$ but also the normal derivative $\frac{\partial}{\partial t} u(t_0, y)$ vanishes for all $y \in \mathbb{R}^d$. Therefore these results could not be used in our setting.

In Section 2 we used the assumption that a polyharmonic function $u(t, y)$ is odd at two points $t_1 < t_2$. This is a rather strong assumption: roughly speaking, we shall prove that this implies that $u(t, y)$ is periodic. In order to prove this we need the following technical result which might be part of mathematical folklore:

Proposition 12 *Suppose that $u(t, y)$ is a polyharmonic function on $(a, b) \times \mathbb{R}^d$ and suppose that there exists a positive constant $c < b - a$ such that*

$$u(t, y) = u(t + c, y) \quad \text{for all } y \in \mathbb{R}^d, \tag{15}$$

and for all $t \in (a, b)$ such that $t + c \in (a, b)$. Then u possesses a polyharmonic extension \tilde{u} defined on $(-\infty, \infty) \times \mathbb{R}^d$ such that (15) holds for \tilde{u} and for all $t \in (-\infty, \infty)$ and $y \in \mathbb{R}^d$.

Proof. Define $V_k := (a + kc, b + kc) \times \mathbb{R}^d$ for each integer k . Define a polyharmonic function u_k on V_k by setting $u_k(x, y) := u(x - kc, y)$. Now we define the extension $\tilde{u}(t, y)$ by setting $\tilde{u}(t, y) = u_k(t, y)$ whenever $(t, y) \in V_k$. We have to show the correctness of the definition of \tilde{u} . Let us suppose that $(t, y) \in V_k$ and $(t, y) \in V_l$

for two different integers k, l . Then $a < t - kc < b$ and $a < t - lc < b$. We may assume that $k < l$. We have to show that

$$u_k(t, y) = u(t - kc, y) = u(t - lc, y) = u_l(t, y).$$

Since $t - kc = t - lc + (l - k)c \in (a, b)$ and $l - k > 0$ and $c > 0$ we infer that

$$t_j := t - lc + jc \in (a, b) \quad \text{for } j = 0, \dots, l - k.$$

From our assumption we infer that $u(t_j, y) = u(t_{j+1}, y)$ for $j = 0, \dots, l - k$, and therefore $u(t_0, y) = u(t_{l-k}, y)$ which is the statement. ■

Theorem 13 *Suppose that $u(t, y)$ is a polyharmonic function on $(a, b) \times \mathbb{R}^d$ which is odd at two different points $t_1 < t_2 \in (a, b)$. If $a < t_1 - \delta$ and $t_2 + \delta < b$ for $\delta := t_2 - t_1$ then u possesses a polyharmonic extension \tilde{u} defined on $(-\infty, \infty) \times \mathbb{R}^d$ satisfying*

$$\tilde{u}(t + 2(t_2 - t_1), y) = \tilde{u}(t, y) \quad \text{for all } (t, y) \in (-\infty, \infty) \times \mathbb{R}^d. \quad (16)$$

Proof. We restrict the function u to the domain $(t_1 - \delta, t_2 + \delta) \times \mathbb{R}^d$. We want to apply Proposition 12 and we check now the validity of its assumption: Let $t \in (t_1 - \delta, t_2 + \delta)$ be given such that $t + 2\delta \in (t_1 - \delta, t_2 + \delta)$. We have to show that $u(t, y) = u(t + 2\delta, y)$. Note that $t + 2\delta < t_2 + \delta = t_1 + 2\delta$, so $t < t_1$, and clearly $t_1 - \delta < t$. Consider at first the point $t_2 + s$ where $s := t + t_2 - 2t_1$. Note that $t_2 + s = t + 2t_2 - 2t_1 = t + 2\delta \in (t_1 - \delta, t_2 + \delta) \subseteq (a, b)$. Further

$$t_2 - s = t_2 - t - t_2 + 2t_1 = 2t_1 - t \in (a, b) \quad (17)$$

since $2t_1 - t < 2t_1 + \delta - t_1 = t_1 + \delta = t_2 < b$ and $2t_1 - t > 2t_1 - t_1 = t_1 > a$. As $u(t, y)$ is odd at t_2 we infer that

$$u(t + 2t_2 - 2t_1, y) = u(t_2 + s, y) = -u(t_2 - s, y) = -u(t_1 - (t_1 - t)).$$

Next we use that $u(t, y)$ is odd at t_1 . Consider $\sigma := t_1 - t$. Formula (17) shows that $t_1 + \sigma = 2t_1 - t \in (a, b)$. As u is odd at t_1 it follows that

$$-u(t_1 - (t_1 - t), y) = u(t, y).$$

By Proposition 12 there exists a polyharmonic extension \tilde{u} of the function u restricted to $(t_1 - \delta, t_2 + \delta) \times \mathbb{R}^d$. Since u and \tilde{u} are polyharmonic functions which agree on $(t_1 - \delta, t_2 + \delta) \times \mathbb{R}^d$ it follows that \tilde{u} is an extension of the polyharmonic function u . Similarly, the equation (16) holds since it is true for the restriction of u to the open set $(t_1 - \delta, t_2 + \delta) \times \mathbb{R}^d$. ■

6 Appendix 2

We provide here a proof for the last statement in the proof of Theorem 9, namely

$$\frac{\tilde{p}_{j,l}(t)}{t} = p_{j,l}(t).$$

Indeed, since $p_{j,0}(t) = \prod_{s=0}^{j-1} ((j-s)\pi - t)$ we see that

$$\begin{aligned} \tilde{p}_{j,0}(t) &= -(-\pi j - t)(\pi(j-1) - t)p_{j-1,0}(t + \pi) \\ &= -(\pi j - t)(\pi(j-1) - t) \prod_{s=0}^{j-2} ((j-2-s)\pi - t) \\ &= t \cdot \prod_{s=0}^{j-1} ((j-s)\pi - t) = t \cdot p_{j,0}(t) \end{aligned}$$

and similarly it follows that $\tilde{p}_{j,j}(t) = t \cdot p_{j,j}(t)$. Next we compute $p_{j-1,l}(t + \pi)$ and $p_{j-1,l-1}(t - \pi)$ in order to compute $\tilde{p}_{j,l}(t)$. Note that

$$\prod_{s=0}^{l-1} ((j-1-s)\pi - (\pi + t))((j-1-s)\pi + \pi + t) = \prod_{s=0}^{l-1} ((j-2-s)\pi - t)((j-s)\pi + t)$$

and

$$\prod_{s=0}^{l-1} ((j-2-s)\pi - t) = \prod_{s=2}^{l+1} ((j-s)\pi - t).$$

We have

$$\prod_{s=0}^{j-2-2l} ((j-1-l-s)\pi - (\pi + t)) = \prod_{s=0}^{j-2-2l} ((j-l-s-2)\pi - t) = \prod_{s=2}^{j-2l} ((j-l-s)\pi - t)$$

It follows that

$$\begin{aligned} A &= (\pi j - t)(\pi(j-1) - t) \cdot p_{j-1,l}(\pi + t) \\ &= \binom{j-1}{l} \prod_{s=0}^{l-1} ((j-s)\pi - t) \prod_{s=0}^{l-1} ((j-s)\pi + t) \prod_{s=2}^{j-2l} ((j-l-s)\pi - t) \end{aligned}$$

and therefore

$$\begin{aligned} A &= \binom{j-1}{l} \prod_{s=0}^{l-1} ((j-s)\pi - t)((j-s)\pi + t) \cdot \prod_{s=0}^{j-2l} ((j-l-s)\pi - t) \\ &= p_{j,l}(t) \frac{j-l}{l} ((j-l-(j-2l))\pi - t) = p_{j,l}(t) \frac{j-l}{l} (l\pi - t). \end{aligned}$$

Next we consider $p_{j-1,l-1}(t - \pi)$. At first we see that

$$\begin{aligned} & \prod_{s=0}^{l-2} ((j-1-s)\pi - (t - \pi)) ((j-1-s)\pi + t - \pi) \\ &= \prod_{s=0}^{l-2} ((j-s)\pi - t) \prod_{s=0}^{l-2} ((j-2-s)\pi + t) = \prod_{s=0}^{l-2} ((j-s)\pi - t) \prod_{s=2}^l ((j-s)\pi + t) \end{aligned}$$

and

$$\prod_{s=0}^{j-2-2(l-1)} ((j-1-(l-1)-s)\pi - (t - \pi)) = \prod_{s=0}^{j-2l} ((j+1-l-s)\pi - t).$$

This implies

$$\begin{aligned} B &= (\pi j + t)(\pi(j-1) + t)p_{j-1,l-1}(t - \pi) \\ &= \binom{j-1}{l-1} \prod_{s=0}^{l-2} ((j-s)\pi - t) \prod_{s=0}^l ((j-s)\pi + t) \prod_{s=0}^{j-2l} ((j+1-l-s)\pi - t) \\ &= \binom{j-1}{l-1} \prod_{s=0}^{l-1} ((j-s)\pi - t) \prod_{s=0}^l ((j-s)\pi + t) \prod_{s=1}^{j-2l} ((j+1-l-s)\pi - t) \\ &= p_{j,l}(t) \frac{l}{j} ((j-l)\pi + t). \end{aligned}$$

It follows that

$$\tilde{p}_{j,l}(t) = -A + B = p_{j,l}(t) \left[\frac{l}{j} ((j-l)\pi + t) - \frac{j-l}{j} (l\pi - t) \right].$$

The term in square brackets is equal to t and the claim is proven.

Acknowledgement 14 *Both authors acknowledge the partial support by the Bulgarian NSF Grant I02/19, 2015. The first named author acknowledges partial support by the Humboldt Foundation.*

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